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Study of Spectral Families and Banach Spaces

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ABSTRACT: The Present paper deals with the study of Banach spaces. To make concepts clear basic study is divided into different parts.

Keywords: Spectral Families, Banach Spaces, Differential Geometry, reflexive Banach space, Differential equation

I. INTRODUCTION

Diestel, Joseph (1975) discussed Support functional for closed bounded convex subsets of a Banach space. Convexity and differentiability of norms. Uniformly convex and uniformly smooth Banach spaces. The classical renorming theorems. Weakly compactly generated banach spaces. The Radon-Nikodým theorem for vector measures. Pandelis Dodos (1993) studied the Standard Borel Space of All Separable Banach Spaces, The Baire Sum, Amalgamated Spaces, Zippin's Embedding Theorem, The Bourgain–Pisier Construction, Strongly Bounded Classes of Banach Spaces. James Stuart Groves (2000) gives stochastic processes in Banach space. David R. Larson *et al.* (1991) use several fundamental results which characterize frames for a Hilbert space to give natural generalizations of Hilbert space frames to general Banach spaces. However, they will see that all of these natural generalizations (as well as the currently used generalizations) are equivalent to properties already extensively developed in Banach space theory. They show that the dilation characterization of framing pairs for a Hilbert spaces generalizes (with much more effort) to the Banach space setting. Finally examine the relationship between frames for Banach spaces and various forms of the Banach space approximation properties.

Fernando Bombal (2000) studied some classes of distinguished subsets of a Banach space in terms of polynomials and their relationship. This allows developing a systematic approach to study polynomial properties on a Banach space. They apply this approach to obtain several known and new results on the symmetric tensor product of a Banach space in a unified way. Stephen Semmes (1988) discussed conceptual approach to banach spaces and deals with many ways of looking at interpolation of banach spaces. In particularly develops Differential Geometry and Differential equation.

Jesus M.F. Castillo (2010) focused on some of the five basic elements of category theory –namely, i) The definition of category, function and natural transformation; ii) Limits and co limits; iii) Adjoint factors; plus a naive presentation of Kan extensions– to support the simplest answer "tools that work and a point of view that helps to understand problems. Homology treated in a second part.

Definition 1.1. A (real) complex normed space is a (real) complex vector space X together with a map : X R, called the norm and denoted $\|\cdot\|$ such that

(i) $\|x\| \geq 0 \, for \; all \; x \in X, and \; \|x\| \; = 0 \; if \; and \; only \; if \; x = 0$
(ii) $\|\alpha x\| = |\alpha|\|x\|, for \; all \; x \in X \; and \; all \; \alpha \in \mathbb{C} \; (or \; \mathbb{R})$
(iii) $\|x + y\| \leq \|x\| + \|y\|, for \; all \; x, y \in X$

Remark 1.2. If in (i) we only require that $\|x\| \geq 0$, for all $x \in X$, then $\|\cdot\|$ is called a seminorm.

Remark 1.3. If X is a normed space with norm $\|\cdot\|$ it is readily checked that the formula $d(x, y) = \|x - y\|$ for $x, y \in X$, defines a metric d on X. Thus a normed space is naturally a metric space and all metric space concepts are meaningful. For example, convergence of sequences in X means convergence with respect to the above metric.

Definition 1.4. A complete normed space is called a Banach space. Thus, a normed space X is a Banach space if every Cauchy sequence in X converges (where X is given the metric space structure as outlined above). One may consider real or complex Banach spaces depending, of course, on whether X is a real or complex linear space.

II. DIFFERENT PART OF STUDIES

Part 1. Let X be a reflexive Banach space and let a ∈L(X). Then A is well-Bounded if and only if there is a spectral family E (·) in X, concentrated on a compact interval J in R, such that

$$A = \int_J^{\oplus} \lambda dE(\lambda)$$

When A is well-bounded, the spectral family E (·) is uniquely determined (and is called the spectral family of A).

There is a version of the non-reflexive case but it involves a weaker notion of spectral family involving projections acting on X and the spectral integrals are interpreted in a weak-star sense. (for a slightly different approach.) Furthermore, the uniqueness assertion is no longer valid. In the present paper, however, we shall mainly consider the reflexive case since the theory is more elegant in that context and encompasses the main examples.

Well-boundedness gives an analogue on Banach spaces of the Hilbert space concept of self-adjointness and suggests a corresponding analogue for the class of unitary operators. An operator U on a Hilbert space H is unitary if and only if it can be written as

$$U = \int_T \omega \mathcal{E}(d\omega)$$

for some self-adjoint spectral measure $\mathcal{E}(\cdot)$ on the Borel subsets of the unit circle. With U represented in this way, let $E(\lambda) = \mathcal{E}(\Gamma_\lambda)\, for\ 0 \leq \lambda \leq 2\pi$ where Γ_λ is the arc $\{e^{t\lambda}: 0 \leq t \leq \lambda\}$, and extend E(·) to R by setting

$$E(\lambda) = 0 \ \ (\lambda < 0), \ \ E(\lambda) = I \ \ (\lambda \geq 2\pi)$$

We obtain a spectral family in H concentrated on [0, 2] and it can be rewritten as

$$U = \int_{[0,2\pi]}^{\oplus} e^{t\lambda} dE(\lambda)$$

This leads us to consider an operator U on a Banach space X which has a representation of the form above equation for some spectral family E(·) in X concentrated on [0, 2π]. Such an operator U is invertible with inverse $\int_{[0,2\pi]}^{\oplus} e^{t\lambda} dE(\lambda)$ and the mapping $f \rightarrow \int_{[0,2\pi]}^{\oplus} f(e^{t\lambda}) dE(\lambda)$ of AC (T) into L(X) is a norm-continuous, identity-preserving algebra homomorphism. In particular, writing q(U) for $\sum_{n\epsilon Z} a_n U^n, where\ q(e^{it}) = \sum_{n\epsilon Z} a_n e^{int}$ is a trigonometric polynomial, we have

$$\|q(U)\| \leq K\|q\|_T$$

Where $K \equiv \sup\{\|E(\lambda)\|: \lambda \in [0,2\pi]\}$

In view of the terminology for well-bounded operators, we shall say that an operator U on a Banach space X is trigonometrically well-bounded if it is invertible and there is a constant K holds for all trigonometric polynomials. The above discussion shows that every operator U with a representation as in trigonometrically well-bounded. Strictly speaking, the existence of such a representation was originally taken as the definition of trigonometrically well-boundedness although the present definition is more natural, given the definition of well-boundedness. However, the two definitions Coincide on reflexive spaces.

Part 2. Let X is a reflexive Banach space and let U∈L(X). Then U is trigonometrically well-bounded if and only if there is a spectral family E(·) in X concentrated on [0,2π] such that

$$U = \int_{[0,2\pi]}^{\oplus} e^{t\lambda} dE(\lambda)$$

When U is trigonometrically well-bounded, the spectral family E (·) can be chosen to be left continuous in the strong operator topology at 2π and, with this normalization, is uniquely determined. (It is then called the spectral decomposition of U.)

Corresponding to the fact that an operator on a Hilbert space is unitary if and only if is of the form e^{iA} for some self-adjoint A; we have the corresponding connection between well-bounded and trigonometrically well-bounded operators.

Part 3. An operator U on a reflexive Banach space X is trigonometrically well-bounded if and only if it is of the form e^{iA} for some well-bounded operator A on X. In this case, A can be chosen to have spectrum contained in [0, 2π] with 2π not an Eigen value and these spectral conditions determine A uniquely. (A is then called the argument of U and is denoted by arg U.)

It is easy to see that, if U is trigonometrically well-bounded with spectral decomposition E(·), then arg

$$U = \int_{[0,2\pi]}^{\oplus} \lambda dE(\lambda)$$

Part 4. Let U be an invertible operator on a Banach space X. Then there exists a constant K such that $\|q(U)\| \le K\|q\|_T$ for all trigonometric polynomials q if and only if

$$sup\{\| \ \sigma_n(U, e^{i\lambda})\|: n\epsilon\mathbb{N} \ \ and \ \lambda \in \mathbb{R}\} < \infty$$

When X is reflexive, it can be replaced by the stronger condition that $\{\ \sigma_n(U, e^{i\lambda})\}$ converges in the strong (or weak) operator topology at each $e^{i\lambda} \in T$ with limit which is uniformly bounded for $e^{i\lambda} \in T$. This is discussed in greater detail.

Part 5. Let U be a power-bounded invertible operator on a UMD space X. Then there is a spectral family in X, concentrated on [0, 2π], such that

$$U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} dE(\lambda)$$

It is worth commenting that closed subspaces of L^p-spaces are UMD if $1 < p < \infty$; in particular, Hilbert spaces belong to the class UMD. Since an invertible power-bounded operator on a Hilbert is similar to a unitary operator Part 5 shows that a vestige of this spectral structure remains for power-bounded operators on general UMD spaces.

Furthermore, it should be remarked that, although the spectral family in Part 5 will not in general generate a spectral measure on the Borel subsets of [0, 2π], nevertheless it does give rise to a spectral measure associated with a dyadic partitioning of [0, 2π] [6] for details). This can be viewed as an operator-theoretic analogue of the classical Littlewood–Paley theorem for L^p (Z) and is underpinned by a version of the

Littlewood– Paley result for L^p (T, X) valid when X is a UMD space, together with several transference arguments.

III. CONCLUSION

Various parts represent recent developments in Banach Spaces with Hilbert Transform; various notations of self adjointness have been developed.

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